

Lax formalism for q -Painlevé equations with affine Weyl group symmetry of type $E_n^{(1)}$

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Abstract. An explicit form of the Lax pair for the q -difference Painlevé equation with affine Weyl group symmetry of type $E_8^{(1)}$ is obtained. Its degeneration to $E_7^{(1)}$, $E_6^{(1)}$ and $D_5^{(1)}$ cases are also given.

Key Words and Phrases: Painlevé equation, Lax formalism, q -difference equation.

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1 Introduction

A Lax formalism for the elliptic difference Painlevé equation [9] was obtained in [14]. The construction is concrete including the specification of the unknown variable (f, g) of the Painlevé equation. The explicit formula of the elliptic Painlevé equation and its Lax from, however, seems to be too complicated to write it down in tidy form.

On the other hand, for the multiplicative (i.e. q -difference) and additive cases, rather concise expressions for the Painlevé equations have been known [8]. The aim of this paper is to adjust the construction of [14] to the q -Painlevé equations in [8] and write down the corresponding Lax equations explicitly. We note that the Lax formulations of difference Painlevé equations were obtained in [1] for additive $E_6^{(1)}$, [2] for additive $E_7^{(1)}$, $E_8^{(1)}$ and [10] for q - $E_6^{(1)}$ cases (see also [11]).

In general, the scalar Lax pair consists of a linear (difference) equation $L_1 = 0$, and its deformation equation $L_2 = 0$. Though the Lax equations L_1, L_2 in [14] were both of degree $(3, 2)$ in variables (f, g) , other choices of equation L_2 are possible depending on the direction of deformation. In this paper, we will give an explicit expressions of the Lax pair L_1, L_2 for q -Painlevé equation (7) corresponding to a distinguished direction. The first equation L_1 (14) is of degree $(3, 2)$ which is the explicit realization of that in [14]. The second equation L_2 (17) is of degree $(1, 1)$ and hence much more economical.

The contents of this paper is as follows. In section 2, the explicit expression of the

q -Painlevé equation of type $E_8^{(1)}$ is recapitulated. (Its Weyl group symmetry is explained in Appendix A.) In section 3, corresponding Lax equations are given in explicit form. The compatibility is proved in section 4. Finally, in section 5, the degeneration to $E_7^{(1)}$, $E_6^{(1)}$ and $D_5^{(1)}$ cases are considered. The Lax form for q - $E_7^{(1)}$ case seems also to be new.

2 Fundamental equations

Let $h_1, h_2, u_1, \dots, u_8$ and $q = h_1^2 h_2^2 / (u_1 \cdots u_8)$ be complex parameters. We consider a configuration of the following eight points in $\mathbb{P}^1 \times \mathbb{P}^1$:

$$P_i = P(u_i), \quad P(u) = (f(u), g(u)), \quad f(u) = u + \frac{h_1}{u}, \quad g(u) = u + \frac{h_2}{u}. \quad (1)$$

The functions $f = f(u), g = g(u)$ give a parametrization of a rational curve $C_0 : \varphi(f, g) = 0$ of degree $(2, 2)$, where

$$\varphi(f, g) = (f - g) \left(\frac{f}{h_1} - \frac{g}{h_2} \right) - (h_1 - h_2) \left(\frac{1}{h_1} - \frac{1}{h_2} \right). \quad (2)$$

We define a polynomial $U(z)$ as

$$U(z) = \prod_{i=1}^8 (z - u_i) = \sum_{i=0}^8 (-1)^i m_{8-i} z^i. \quad (3)$$

Hence $m_0 = 1, m_8 = h_1^2 h_2^2 / q$. We also define $P_n(h, x), P_d(h, x)$ by

$$\begin{aligned} \left(z - \frac{h}{z}\right) P_n(h, z + \frac{h}{z}) &= \frac{1}{z^3} U(z) - \left(\frac{z}{h}\right)^3 U\left(\frac{h}{z}\right), \\ \left(z - \frac{h}{z}\right) P_d(h, z + \frac{h}{z}) &= z^5 U\left(\frac{h}{z}\right) - \left(\frac{h}{z}\right)^5 U(z). \end{aligned} \quad (4)$$

Then they satisfies

$$\begin{aligned} P_d(h, z + \frac{h}{z}) + h^3 z^2 P_n(h, z + \frac{h}{z}) - \left(\frac{h}{z}\right)^3 \left(z + \frac{h}{z}\right) U(z) &= 0, \\ P_d(h, g) &= h^4 P_n\left(\frac{1}{h}, \frac{g}{h}\right) \Big|_{m_i \mapsto m_{8-i}}. \end{aligned} \quad (5)$$

Explicitly, we have

$$\begin{aligned} P_n(h, g) &= m_0 g^4 - m_1 g^3 + (m_2 - 3hm_0 - h^{-3}m_8)g^2 \\ &\quad + (2hm_1 - m_3 + h^{-2}m_7)g + (h^2m_0 - hm_2 + m_4 - h^{-1}m_6 + h^{-2}m_8), \\ P_d(h, g) &= m_8 g^4 - hm_7 g^3 + (h^2m_6 - 3hm_8 - h^5m_0)g^2 \\ &\quad + (2h^2m_7 - h^3m_5 + h^5m_1)g + (h^6m_0 - h^5m_2 + h^4m_4 - h^3m_6 + h^2m_8). \end{aligned} \quad (6)$$

The q -Painlevé equation of type $E_8^{(1)}$ can be described by the following bi-rational transformations [8]:

$$T : (h_1, h_2, u_1, \dots, u_8; f, g) \mapsto \left(\frac{h_1}{q}, h_2q, u_1, \dots, u_8; \bar{f}, \bar{g}\right), \quad (7)$$

where

$$\frac{(\bar{f} - g)(f - g) - (\frac{h_1}{q} - h_2)(h_1 - h_2)\frac{1}{h_2}}{(\frac{\bar{f}q}{h_1} - \frac{g}{h_2})(\frac{f}{h_1} - \frac{g}{h_2}) - (\frac{q}{h_1} - \frac{1}{h_2})(\frac{1}{h_1} - \frac{1}{h_2})h_2} = \frac{h_1^2 h_2^4 P_n(h_2, g)}{q P_d(h_2, g)}, \quad (8)$$

and

$$\frac{(\bar{f} - \bar{g})(\bar{f} - g) - (\frac{h_1}{q} - h_2q)(\frac{h_1}{q} - h_2)\frac{q}{h_1}}{(\frac{\bar{f}q}{h_1} - \frac{\bar{g}}{h_2q})(\frac{\bar{f}q}{h_1} - \frac{g}{h_2}) - (\frac{q}{h_1} - \frac{1}{h_2q})(\frac{q}{h_1} - \frac{1}{h_2})\frac{h_1}{q}} = \frac{h_1^4 h_2^2 P_n(\frac{h_1}{q}, \bar{f})}{q^3 P_d(\frac{h_1}{q}, \bar{f})}. \quad (9)$$

Under the transformation

$$f \leftrightarrow \bar{g}, \quad g \leftrightarrow \bar{f}, \quad h_1 \rightarrow h_2q, \quad h_2 \rightarrow \frac{h_1}{q}, \quad (10)$$

the equations (8) and (9) are transformed with each other.

Define polynomial $V = V(f_0, f)$ as

$$\begin{aligned} V(f_0, f) = & q \left[(f_0 - g)(f - g) - (\frac{h_1}{q} - h_2)(h_1 - h_2)\frac{1}{h_2} \right] P_d(h_2, g) \\ & - h_1^2 h_2^4 \left[(\frac{f_0q}{h_1} - \frac{g}{h_2})(\frac{f}{h_1} - \frac{g}{h_2}) - (\frac{q}{h_1} - \frac{1}{h_2})(\frac{1}{h_1} - \frac{1}{h_2})h_2 \right] P_n(h_2, g). \end{aligned} \quad (11)$$

Then eq.(8) is written as

$$V(\bar{f}, f) = 0. \quad (12)$$

In the following, we also use the notation

$$\bar{f}(u) = u + \frac{h_1}{qu}, \quad \bar{g}(u) = u + \frac{h_2q}{u}. \quad (13)$$

3 Lax equations

Let us define the Lax pair for the q -Painlevé equations (8), (9). The first equation $L_1 = 0$ is a three term q -difference equation for $Y(\frac{z}{q}), Y(z), Y(qz)$ defined by

$$\begin{aligned} L_1 = & \frac{q^5 U(\frac{z}{q})}{(z^2 - h_1 q^2) \{f - f(\frac{z}{q})\}} \left[Y(\frac{z}{q}) - \frac{g - g(\frac{h_1 q}{z})}{g - g(\frac{z}{q})} Y(z) \right] \\ & + \frac{z^8 U(\frac{h_1}{z})}{(z^2 - h_1) h_1^4 \{f - f(z)\}} \left[Y(qz) - \frac{g - g(z)}{g - g(\frac{h_1}{z})} Y(z) \right] \\ & + \frac{(h_1 - h_2) z^2 (z^2 - h_1 q) V(\bar{f}(\frac{z}{q}), f)}{h_1^3 h_2^3 q g \varphi \{g - g(\frac{h_1}{z})\} \{g - g(\frac{z}{q})\}} Y(z), \end{aligned} \quad (14)$$

where V is in eq.(11). The following proposition shows that the equation $L_1 = 0$ has the geometric properties described in [14].

Proposition 3.1 *Put $F(f, g) = \varphi\{f - f(\frac{z}{q})\}\{f - f(z)\}L_1$. Then the algebraic curve $F = 0$ in $\mathbb{P}^1 \times \mathbb{P}^1$ satisfy the following properties:*

- (i) *It is of degree (3, 2).*
- (ii) *It passes the 12 points $P_1, \dots, P_8, P(z), P(\frac{h_1 q}{z}), Q(z)$ and $Q(\frac{z}{q})$, where $Q(u) = (f, g)$ is defined by $f = f(u)$ and $\frac{g - g(u)}{g - g(\frac{h_1}{u})} = \frac{Y(qu)}{Y(u)}$, for $u = z, \frac{z}{q}$.*

Moreover, these conditions determine the curve $F = 0$ uniquely.

Proof. (i) It is easy to see that the coefficients of $Y(qz)$ and $Y(\frac{z}{q})$ in F are polynomials in (f, g) of degree (3, 2). The coefficient of $Y(z)$ looks a rational function with numerator of degree (3, 6) and denominator of degree (0, 3). However its residues at $g = 0$, $g = g(\frac{z}{q})$, $g = g(\frac{h_1}{z})$ and leading term at $g \rightarrow \infty$ are proportional to

$$\begin{aligned} &P_d(h_2, 0) - h_2^4 P_n(h_2, 0), && \text{at } g = 0 \\ &P_d(h_2, g(\frac{z}{q})) + h_2^3 (\frac{z}{q})^2 P_n(h_2, g(\frac{z}{q})) - (\frac{h_2 q}{z})^3 g(\frac{z}{q}) U(\frac{z}{q}), && \text{at } g = g(\frac{z}{q}) \\ &P_d(h_2, g(\frac{h_1}{z})) + h_2^3 (\frac{h_1}{z})^2 P_n(h_2, g(\frac{h_1}{z})) - (\frac{h_2 z}{h_1})^3 g(\frac{h_1}{z}) U(\frac{h_1}{z}), && \text{at } g = g(\frac{h_1}{z}) \\ &(h_1^2 h_2^2 m_0 - q m_8) g^3, && \text{at } g \rightarrow \infty \end{aligned} \tag{15}$$

and all of them are zero due to eqs (5), (6). Hence the property (i) is proved.

(ii) The coefficients of $Y(qz)$ and $Y(\frac{z}{q})$ of polynomial F trivially vanish at $P_1, \dots, P_8, P(z)$ and $P(\frac{h_1 q}{z})$. The coefficient of $Y(z)$ is proportional to

$$(u - z)(h_1 q - uz)\{P_d(h_2, g(u)) + h_2^3 u^2 P_n(h_2, g(u))\} = (u - z)(h_1 q - uz)(\frac{h_2}{u})^3 g(u) U(u) \tag{16}$$

under the specialization $f = f(u)$ and $g = g(u)$. Hence, it also vanishes at $P_1, \dots, P_8, P(z)$ and $P(\frac{h_1 q}{z})$. The vanishing of F at $Q(z)$ and $Q(\frac{z}{q})$ can be directly seen from the structure of L_1 . The property (ii) is proved.

The uniqueness follows from simple dimensional argument. \square

In [14], the second Lax equation $L_2 = 0$ is also defined as a curve of degree (3, 2) with similar vanishing conditions. However, the L_2 equation (the deformation equation) depends on the direction in $E_8^{(1)}$ -translations and the one in [14] is not what we want here. In fact, we can take more simple Lax equation $L_2 = 0$ given by

$$L_2 = \{g - g(\frac{z}{q})\}Y(\frac{z}{q}) - \{g - g(\frac{h_1 q}{z})\}Y(z) + \{f - f(z)\}(\frac{h_1}{z} - \frac{z}{q^2})\bar{Y}(\frac{z}{q}). \tag{17}$$

The following is the main result of this paper.

Theorem 3.2 *The compatibility of the equations $L_1 = 0$ (14) and $L_2 = 0$ (17) gives the Painlevé equation (8), (9).*

4 Proof of the main theorem

Here, we will prove the Theorem.3.2.

Using $L_2 = 0$ (17) and its shift $L_2|_{z \rightarrow qz} = 0$, eliminate $Y(qz)$ and $Y(\frac{z}{q})$ from $L_1 = 0$ (14), we get

$$\begin{aligned} L_1 = & \frac{q^3 U(\frac{z}{q})}{z\{g - g(\frac{z}{q})\}} \bar{Y}(\frac{z}{q}) - \frac{z^7 U(\frac{h_1}{z})}{h_1^4 q \{g - g(\frac{h_1}{z})\}} \bar{Y}(z) \\ & + \frac{(h_1 - h_2) z^2 (z^2 - h_1 q) V(\bar{f}(\frac{z}{q}), f)}{h_1^3 h_2^3 q g \varphi \{g - g(\frac{h_1}{z})\} \{g - g(\frac{z}{q})\}} Y(z) = 0. \end{aligned} \quad (18)$$

We introduce auxiliary variable $W(z)$ by

$$W(\frac{z}{q}) = \bar{Y}(\frac{z}{q}) - \frac{z^8}{h_1^4 q^4} \frac{\{g - g(\frac{z}{q})\}}{\{g - g(\frac{h_1}{z})\}} \frac{U(\frac{h_1}{z})}{U(\frac{z}{q})} \bar{Y}(z), \quad (19)$$

then, from eq. (18) we have

$$\frac{q^3 U(\frac{z}{q})}{z\{g - g(\frac{z}{q})\}} W(\frac{z}{q}) + \frac{(h_1 - h_2) z^2 (z^2 - h_1 q) V(\bar{f}(\frac{z}{q}), f)}{h_1^3 h_2^3 q g \varphi \{g - g(\frac{h_1}{z})\} \{g - g(\frac{z}{q})\}} Y(z) = 0. \quad (20)$$

From eq.(20), $(20)|_{z \rightarrow qz}$ and $L_2|_{z \rightarrow qz}$, we eliminate $Y(z)$ and $Y(qz)$ and obtain

$$\begin{aligned} W(\frac{z}{q}) + & \frac{(h_1 - h_2)(h_1 - z^2)(h_1 q - z^2)(f - f(z)) V(\bar{f}(\frac{z}{q}), f) z^2}{g h_1^3 h_2^3 \varphi q^5 (g - g(\frac{h_1}{z}))(g - g(z)) U(\frac{z}{q})} \bar{Y}(z) \\ & + \frac{(h_1 q - z^2)(g - g(\frac{h_1}{qz})) U(z) V(\bar{f}(\frac{z}{q}), f)}{q^4 (qz^2 - h_1)(g - g(z)) U(\frac{z}{q}) V(\bar{f}(z), f)} W(z) = 0. \end{aligned} \quad (21)$$

Lemma 4.1 *We have*

$$\frac{(f - x_1) V(x_2, f)}{(\bar{f} - x_2) V(\bar{f}, x_1)} = \frac{(h_1 - h_2 q) \varphi}{(h_1 - h_2) \varphi_u}, \quad (22)$$

where $\varphi_u = \varphi|_{\{f \rightarrow \bar{f}, h_1 \rightarrow \frac{h_1}{q}\}}$.

Proof. The (LHS) of eq.(22) is fractional liner function both in x_1 and x_2 . Whose zeros and poles cancel due to eq.(12) hence it is constant. Taking a limit $x_1, x_2 \rightarrow \infty$, the constant is given by

$$\frac{(f - g) P_d(h_2, g) - h_2^3 (f h_2 - g h_1) P_n(h_2, g)}{(\bar{f} - g) P_d(h_2, g) - h_2^3 (\bar{f} h_2 - g h_1 / q) P_n(h_2, g)}. \quad (23)$$

From eq.(8), this coincide with the (RHS) of eq.(22). \square

By using eq.(22), eq.(21) can be written as

$$\begin{aligned} W(\frac{z}{q}) + \frac{(h_1 - h_2q)(h_1 - z^2)(h_1q - z^2)(\bar{f} - \bar{f}(\frac{z}{q}))V(\bar{f}, f(z))z^2}{gh_1^3h_2^3\varphi_uq^5(g - g(\frac{h_1}{z}))(g - g(z))U(\frac{z}{q})}\bar{Y}(z) \\ + \frac{(h_1q - z^2)(\bar{f} - \bar{f}(\frac{z}{q}))(g - g(\frac{h_1}{qz}))U(z)}{q^4(qz^2 - h_1)(\bar{f} - \bar{f}(z))(g - g(z))U(\frac{z}{q})}W(z) = 0. \end{aligned} \quad (24)$$

Using eq.(19), express W in terms of Y , we finally obtain

$$\begin{aligned} L_{1u} = \frac{U(\frac{z}{q})}{(z^2 - h_1q^2)\{\bar{f} - \bar{f}(\frac{z}{q})\}} \left[\bar{Y}(\frac{z}{q}) - \frac{z^8}{h_1^4q^4} \frac{U(\frac{h_1}{z})}{U(\frac{z}{q})} \frac{g - g(\frac{z}{q})}{g - g(\frac{h_1}{z})} \bar{Y}(z) \right] \\ + \frac{z^8U(\frac{h_1}{qz})}{(qz^2 - h_1)h_1^4\{\bar{f} - \bar{f}(z)\}} \left[\bar{Y}(qz) - \frac{h_1^4}{q^4z^8} \frac{U(z)}{U(\frac{h_1}{qz})} \frac{g - g(\frac{h_1}{qz})}{g - g(z)} \bar{Y}(z) \right] \\ + \frac{(h_1 - h_2q)z^2(z^2 - h_1)V(\bar{f}, f(z))}{h_1^3h_2^3q^5g\varphi_u\{g - g(\frac{h_1}{z})\}\{g - g(z)\}} \bar{Y}(z) = 0. \end{aligned} \quad (25)$$

This is the three term difference equation for $\bar{Y}(qz), \bar{Y}(z), \bar{Y}(\frac{z}{q})$ which should be compared with T -evolution of the eq. L_1 for the compatibility. To do this, one should make further variable change from (\bar{f}, g) to (\bar{f}, \bar{g}) , where g and \bar{g} are related by eq.(9). Though the explicit computation of this variable change is rather complicated, one can bypass it by using the geometric method as [14].

Lemma 4.2 *The algebraic curve $\varphi\{\bar{f} - \bar{f}(\frac{z}{q})\}\{\bar{f} - \bar{f}(z)\}L_{1u} = 0$ in variables $(\bar{f}, g) \in \mathbb{P}^1 \times \mathbb{P}^1$ is uniquely characterized by the following properties:*

- (i) *It is of degree (3, 2).*
- (ii) *It passes the 10 points $(\bar{f}, g) = (\bar{f}(u), g(u))$ with $u = u_1, \dots, u_8, \frac{z}{q}, \frac{h_1}{qz}$, and two more points defined by $\bar{f} = \bar{f}(u)$ and $\frac{q^4u^8}{h_1^4} \frac{U(\frac{h_1}{qu})}{U(u)} \frac{g - g(u)}{g - g(\frac{h_1}{qu})} = \frac{\bar{Y}(u)}{\bar{Y}(qu)}$, for $u = z, \frac{z}{q}$.*

Proof. Since the structure of eq.(25) is almost the same as eq.(14), the lemma can be shown in the same way as Proposition.3.1. \square

Lemma 4.3 *The bi-rational transformation $\bar{g} = \bar{g}(\bar{f}, g)$ [its inverse $g = g(\bar{f}, \bar{g})$, resp.] is uniquely characterized by the following properties. (i) It is given by a ratio of polynomials of degree (4, 1) passing through the 8 points $(\bar{f}, g) = (\bar{f}(u_i), g(u_i))$ [$(\bar{f}, \bar{g}) = (\bar{f}(u_i), \bar{g}(u_i))$, resp.]. (ii) For generic parameter u , it transforms as*

$$(\bar{f}, g) = (\bar{f}(\frac{h_1}{qu}), g(\frac{h_1}{qu})) \leftrightarrow (\bar{f}, \bar{g}) = (\bar{f}(u), \bar{g}(u)). \quad (26)$$

Proof. The transformation $\bar{g} = \bar{g}(\bar{f}, g)$ [$g = g(\bar{f}, \bar{g})$] is determined by eq.(9), i.e.

$$\begin{aligned} & q^3 \left[(\bar{f} - \bar{g})(\bar{f} - g) - \left(\frac{h_1}{q} - h_2 q\right) \left(\frac{h_1}{q} - h_2\right) \frac{q}{h_1} \right] P_d\left(\frac{h_1}{q}, \bar{f}\right) \\ & - h_1^4 h_2^2 \left[\left(\frac{\bar{f}q}{h_1} - \frac{\bar{g}}{h_2 q}\right) \left(\frac{\bar{f}q}{h_1} - \frac{g}{h_2}\right) - \left(\frac{q}{h_1} - \frac{1}{h_2 q}\right) \left(\frac{q}{h_1} - \frac{1}{h_2}\right) \frac{h_1}{q} \right] P_n\left(\frac{h_1}{q}, \bar{f}\right) = 0. \end{aligned} \quad (27)$$

This equation is linear in both \bar{g} and g , and of degree 4 in \bar{f} since degree 5 terms are cancelled. Hence $\bar{g}(\bar{f}, g)$ [$g(\bar{f}, \bar{g})$] is of degree (4,1). By a similar computation as eq.(16), the eq.(27) is written as

$$\left\{g - g\left(\frac{h_1}{qu}\right)\right\} \left\{q^3 P_d\left(\frac{h_1}{q}, \bar{f}(u)\right) + h_1^3 u^2 P_n\left(\frac{h_1}{q}, \bar{f}(u)\right)\right\} = \left\{g - g\left(\frac{h_1}{qu}\right)\right\} \left(\frac{h_1}{u}\right)^3 \bar{f}(u) U(u) = 0, \quad (28)$$

when $\bar{f} = \bar{f}(u)$ and $\bar{g} = \bar{g}(u)$. Then it follows that (1) if $\bar{f} = \bar{f}(u_i)$ then $\bar{g} = \bar{g}(u_i)$ (regardless of g) and (2) if $\bar{f} = \bar{f}(u)$ and $\bar{g} = \bar{g}(u)$ then $g = g\left(\frac{h_1}{qu}\right)$ for $u \neq u_i$. Hence the properties (i), (ii) are verified. Finally the uniqueness follows from dimensional argument. \square

Lemma 4.4 *Let $F(\bar{f}, g) = 0$ be any curve of degree (3, 2) passing through the 10 points in (ii) of Lemma.4.2. Then the curve in variables (\bar{f}, \bar{g}) obtained from $F(\bar{f}, g(\bar{f}, \bar{g})) = 0$ is also of degree (3, 2) and passes the 10 points $(\bar{f}, \bar{g}) = (\bar{f}(u), \bar{g}(u))$ with $u = u_1, \dots, u_8, z, \frac{h_1}{z}$.*

Proof. From lemma 4.3, we have $g = \frac{A_{41}}{B_{41}}$ where A_{41}, B_{41} are polynomials of degree (4, 1) in (\bar{f}, \bar{g}) vanishing at the 8 points: $(\bar{f}(u_i), \bar{g}(u_i))_{i=1}^8$. Substitute this into $F(\bar{f}, g)$ we have $F(\bar{f}, \frac{A_{41}}{B_{41}}) = \frac{P_{11,2}}{(B_{41})^2}$, where $P_{11,2}(\bar{f}, \bar{g})$ is a polynomial of degree (11, 2) vanishing at the 8 points with multiplicity 2. Since $F(\bar{f}, g(\bar{f}, \bar{g}))|_{\bar{f}=\bar{f}(u_i)} = F(\bar{f}(u_i), g(u_i)) = 0$ (regardless of \bar{g}), the polynomial $P_{11,2}$ is factorized as $P_{11,2} = \tilde{F}_{32}(\bar{f}, \bar{g}) \prod_{i=1}^8 \{\bar{f} - \bar{f}(u_i)\}$ and $\tilde{F}_{32} = 0$ gives the desired curve of degree (3, 2) passing through the 8 points.¹ The other two vanishing conditions follow from the relation $\tilde{F}_{32}(\bar{f}(u), \bar{g}(u)) = 0 \Leftrightarrow F(\bar{f}(u) = \bar{f}(\frac{h_1}{qu}), g(\frac{h_1}{qu})) = 0$ which holds for generic $u (\neq u_i)$. \square

Lemma 4.5 *The equation $L_{1u} = 0$ gives a curve of degree (3, 2) in (\bar{f}, \bar{g}) passing through the 10 points $(\bar{f}, \bar{g}) = (\bar{f}(u), \bar{g}(u))$ with $u = u_1, \dots, u_8, z, \frac{h_1}{z}$.*

Proof. It follows from the Lemma 4.2 and Lemma 4.4. \square

Lemma 4.5 ensures the vanishing properties of L_{1u} which must be satisfied by T -evolution of L_1 at the first 10 points. In order to prove the main theorem, it is enough to prove the following

¹ In terms of the Picard lattice, this part corresponds to the fact that $3H_1 + 2H_2 - \sum_{i=1}^8 E_i$ is invariant under the substitution $H_2 \mapsto 4H_1 + H_2 - \sum_{i=1}^8 E_i$, $E_i \mapsto H_1 - E_i$ ($i = 1, \dots, 8$).

Lemma 4.6 *The equation L_{1u} vanishes at $\overline{Q}(z)$ and $\overline{Q}(\frac{z}{q})$, where $\overline{Q}(u) = (\overline{f}, \overline{g})$ is defined by $\overline{f} = \overline{f}(u)$ and $\frac{\overline{g} - \overline{g}(u)}{\overline{g} - \overline{g}(\frac{h_1}{qu})} = \frac{\overline{Y}(qu)}{\overline{Y}(u)}$, for $u = z, \frac{z}{q}$.*

Proof. When $\overline{f} = \overline{f}(\frac{z}{q})$, from eq.(9) and eq.(5), we have

$$\begin{aligned} \frac{g - g(\frac{z}{q})}{g - g(\frac{h_1}{z})} &= \frac{\overline{g} - \overline{g}(\frac{h_1}{z})}{\overline{g} - \overline{g}(\frac{z}{q})} \frac{h_1 q}{z^2} \frac{P_d(\frac{h_1}{q}, \overline{f}(\frac{z}{q})) + (\frac{h_1}{q})^3 (\frac{z}{q})^2 P_n(\frac{h_1}{q}, \overline{f}(\frac{z}{q}))}{P_d(\frac{h_1}{q}, \overline{f}(\frac{z}{q})) + (\frac{h_1}{q})^3 (\frac{h_1}{z})^2 P_n(\frac{h_1}{q}, \overline{f}(\frac{z}{q}))} \\ &= \frac{\overline{g} - \overline{g}(\frac{h_1}{z})}{\overline{g} - \overline{g}(\frac{z}{q})} \frac{h_1^4 q^4}{z^8} \frac{U(\frac{z}{q})}{U(\frac{h_1}{z})}. \end{aligned} \quad (29)$$

Then, from the residue of eq.(25) at $\overline{f} = \overline{f}(\frac{z}{q})$, we have

$$\frac{\overline{Y}(\frac{z}{q})}{\overline{Y}(z)} = \frac{z^8}{h_1^4 q^4} \frac{U(\frac{h_1}{z})}{U(\frac{z}{q})} \frac{g - g(\frac{z}{q})}{g - g(\frac{h_1}{z})} = \frac{\overline{g} - \overline{g}(\frac{h_1}{z})}{\overline{g} - \overline{g}(\frac{z}{q})}. \quad (30)$$

This is the desired relation for $u = \frac{z}{q}$. The relation for $u = z$ is similar. \square

The proof of the main theorem is completed.

5 Degenerations

In this section, we consider the degeneration limit of the $E_8^{(1)}$ system to $E_7^{(1)}$, $E_6^{(1)}$, $D_5^{(1)}$. For the corresponding q -Painlevé equations see [5], [12] and references therein.

5.1 Degeneration from $E_8^{(1)}$ to $E_7^{(1)}$

We put $(h_1, h_2) = (t\epsilon, \frac{\epsilon}{t})$ and $(u_1, \dots, u_8) = (b_1, \dots, b_4, \frac{\epsilon}{b_5}, \dots, \frac{\epsilon}{b_8})$, hence $q = \frac{b_5 b_6 b_7 b_8}{b_1 b_2 b_3 b_4}$. Under the limit $\epsilon \rightarrow 0$ we have

$$\begin{aligned} P_n(h_2, g) &\rightarrow g^4 B_1(\frac{1}{g}), & P_d(h_2, g) &\rightarrow \epsilon^4 \frac{g^4}{q} B_2(\frac{1}{tg}), \\ P_n(\frac{h_1}{q}, \overline{f}) &\rightarrow \overline{f}^4 B_1(\frac{1}{\overline{f}}), & P_d(\frac{h_1}{q}, \overline{f}) &\rightarrow \epsilon^4 \frac{\overline{f}^4}{q} B_2(\frac{t}{q\overline{f}}), \\ U(\frac{z}{q}) &\rightarrow (\frac{z}{q})^8 B_1(\frac{q}{z}), & U(\frac{h_1}{z}) &\rightarrow \epsilon^4 \frac{1}{q} B_2(\frac{t}{z}), \end{aligned} \quad (31)$$

where

$$B_1(z) = \prod_{i=1}^4 (1 - b_i z), \quad B_2(z) = \prod_{i=5}^8 (1 - b_i z). \quad (32)$$

In addition to this limiting procedure, we also make a change of coordinate : $g \rightarrow \frac{1}{g}$. Then the configuration of the 8 points are $(f, g) = (b_i, \frac{1}{b_i})_{i=1, \dots, 4}$, and $(b_i t, \frac{t}{b_i})_{i=5, \dots, 8}$. They are on the curve $(fg - 1)(fg - t^2) = 0$.

The q -Painlevé equation of type $E_7^{(1)}$ is then given by $(b_i, t, f, g) \mapsto (b_i, \frac{t}{q}, \bar{f}, \bar{g})$, where

$$\begin{aligned} \frac{(fg - 1)(\bar{f}g - 1)}{(fg - t^2)(\bar{f}gq - t^2)} &= \frac{B_1(g)}{t^4 B_2(\frac{g}{t})}, \\ \frac{(\bar{f}g - 1)(\bar{f}\bar{g} - 1)}{(\bar{f}g - t^2)(\bar{f}\bar{g}q^2 - t^2)} &= \frac{B_1(\frac{1}{\bar{f}})}{q^3 B_2(\frac{t}{\bar{f}q})}. \end{aligned} \quad (33)$$

The Lax pair is

$$\begin{aligned} L_1 = & \frac{(1 - t^2)}{gz^2} \left[\frac{qB_1(g)}{(fg - 1)(gz - q)} - \frac{t^4 B_2(\frac{g}{t})}{(fg - t^2)(gz - t^2)} \right] Y(x) \\ & + \frac{B_2(\frac{t}{z})}{t^2(f - z)} \left[Y(qz) - \frac{t^2(1 - gz)}{t^2 - gz} Y(z) \right] \\ & + \frac{t^2 B_1(\frac{g}{z})}{q(fq - z)} \left[Y(\frac{z}{q}) - \frac{qt^2 - gz}{t^2(q - gz)} Y(z) \right] = 0, \end{aligned} \quad (34)$$

and

$$L_2 = \frac{qt^2 - gz}{t^2} Y(z) + (gz - q) Y(\frac{z}{q}) + \frac{gz(fq - z)}{q^2} \bar{Y}(\frac{z}{q}) = 0. \quad (35)$$

5.2 Degeneration from $E_7^{(1)}$ to $E_6^{(1)}$

Degeneration from $E_7^{(1)}$ to $E_6^{(1)}$ is obtained by putting $b_5 \rightarrow b_5/\epsilon$, $b_6 \rightarrow b_6/\epsilon$, $b_7 \rightarrow b_7\epsilon$, $b_8 \rightarrow b_8\epsilon$ and $t \rightarrow t\epsilon$ and taking the limit $\epsilon \rightarrow 0$.

The 8 points configuration: $(f, g) = (b_i, \frac{1}{b_i})_{i=1, \dots, 4}$, $(b_i t, 0)_{i=5, 6}$ and $(0, \frac{t}{b_i})_{i=7, 8}$.

The q -Painlevé equation:

$$\begin{aligned} \frac{(fg - 1)(\bar{f}g - 1)}{f\bar{f}} &= \frac{q(b_1g - 1)(b_2g - 1)(b_3g - 1)(b_4g - 1)}{b_5b_6(b_7g - t)(b_8g - t)}, \\ \frac{(\bar{f}g - 1)(\bar{f}\bar{g} - 1)}{g\bar{g}} &= \frac{q^2(b_1 - \bar{f})(b_2 - \bar{f})(b_3 - \bar{f})(b_4 - \bar{f})}{(\bar{f}q - b_5t)(\bar{f}q - b_6t)}. \end{aligned} \quad (36)$$

The Lax pair:

$$\begin{aligned}
L_1 &= \frac{(b_1q - z)(b_2q - z)(b_3q - z)(b_4q - z)t^2}{q(fq - z)z^4} \left[Y\left(\frac{z}{q}\right) - \frac{gz}{t^2(gz - q)} Y(z) \right] \\
&\quad + \left[\frac{(b_1g - 1)(b_2g - 1)(b_3g - 1)(b_4g - 1)q}{g(fg - 1)z^2(gz - q)} - \frac{b_5b_6(b_7g - t)(b_8g - t)}{fgz^3} \right] Y(z) \\
&\quad + \frac{(b_5t - z)(b_6t - z)}{(f - z)z^2t^2} \left[Y(qz) - \frac{(gz - 1)t^2}{gz} Y(z) \right] = 0, \\
L_2 &= \frac{gz}{t^2} Y(z) + (q - gz) Y\left(\frac{z}{q}\right) - \frac{gz(fq - z)}{q^2} \overline{Y}\left(\frac{z}{q}\right) = 0.
\end{aligned} \tag{37}$$

A 2×2 matrix Lax formalism for the q -Painlevé equation with $E_6^{(1)}$ -symmetry has been obtained by Sakai [10]. The scalar Lax equation here may be equivalent to Sakai's one, though we could not confirm it so far.

5.3 Degeneration from $E_6^{(1)}$ to $D_5^{(1)}$

Degeneration from $E_6^{(1)}$ to $D_5^{(1)}$ is obtained by putting $b_1 \rightarrow b_1/\epsilon$, $b_2 \rightarrow b_2/\epsilon$, $b_3 \rightarrow b_3\epsilon$, $b_4 \rightarrow b_4\epsilon$, $f \rightarrow f\epsilon$, $g \rightarrow g\epsilon$, $t \rightarrow t\epsilon$, $z \rightarrow z\epsilon$, $\overline{Y} \rightarrow \epsilon^{-3}\overline{Y}$, and taking the limit $\epsilon \rightarrow 0$.

The 8 points configuration: $(f, g) = (\infty, \frac{1}{b_i})_{i=1,2}$, $(b_i, \infty)_{i=3,4}$, $(b_it, 0)_{i=5,6}$ and $(0, \frac{t}{b_i})_{i=7,8}$.
The q -Painlevé equation:

$$\begin{aligned}
f\overline{f} &= \frac{b_5b_6(b_7g - t)(b_8g - t)}{q(b_1g - 1)(b_2g - 1)}, \\
g\overline{g} &= \frac{(\overline{f}q - b_5t)(\overline{f}q - b_6t)}{q^2b_1b_2(b_3 - \overline{f})(b_4 - \overline{f})}.
\end{aligned} \tag{38}$$

The Lax pair:

$$\begin{aligned}
L_1 &= \frac{b_1b_2q(b_3q - z)(b_4q - z)t^2}{(fq - z)z^2} \left[Y\left(\frac{z}{q}\right) + \frac{qz}{qt^2} Y(z) \right] \\
&\quad + \left[\frac{(b_1g - 1)(b_2g - 1)}{g} - \frac{b_5b_6(b_7g - t)(b_8g - t)}{fgz} \right] Y(z) \\
&\quad + \frac{(b_5t - z)(b_6t - z)}{(f - z)t^2} \left[Y(qz) + \frac{t^2}{gz} Y(z) \right] = 0, \\
L_2 &= \frac{gz}{t^2} Y(z) + qY\left(\frac{z}{q}\right) - \frac{gz(fq - z)}{q^2} \overline{Y}\left(\frac{z}{q}\right) = 0.
\end{aligned} \tag{39}$$

This result is essentially equivalent to the original 2×2 construction by Jimbo-Sakai [3]. Further degenerations of this has been comprehensively studied by Murata [7]. The Lax formalism of Jimbo-Sakai is also derived from q -KP/UC hierarchy, and this method is also effective to the higher order generalizations [13].

A Weyl group actions

Here, we will discuss the affine Weyl group symmetry of the q -Painlevé equation (8) (9), based on the constructions in [4],[6] and [12].

Define multiplicative transformations s_{ij} , c , μ_{ij} , ν_{ij} ($1 \leq i \neq j \leq 8$) acting on variables $h_1, h_2, u_1, \dots, u_8$ as

$$\begin{aligned} s_{ij} &= \{u_i \leftrightarrow u_j\}, & c &= \{h_1 \leftrightarrow h_2\}, \\ \mu_{ij} &= \{h_1 \mapsto \frac{h_1 h_2}{u_i u_j}, & u_i &\mapsto \frac{h_2}{u_j}, & u_j &\mapsto \frac{h_2}{u_i}\}, \\ \nu_{ij} &= \{h_2 \mapsto \frac{h_1 h_2}{u_i u_j}, & u_i &\mapsto \frac{h_1}{u_j}, & u_j &\mapsto \frac{h_1}{u_i}\}. \end{aligned} \quad (40)$$

These actions generate the affine Weyl group of type $E_8^{(1)}$. A choice of simple reflections is

$$\begin{array}{ccccccc} & & s_{12} & & & & \\ & & | & & & & \\ c & - & \mu_{12} & - & s_{23} & - & s_{34} & - & \cdots & - & s_{78} & . \end{array} \quad (41)$$

The transformations (40) naturally act on the 8 points configuration $(f_i, g_i) = (u_i + \frac{h_1}{u_i}, u_i + \frac{h_2}{u_i})$ as $\mu_{12}(f_1) = \frac{h_1}{u_1} + \frac{h_2}{u_2}$, $\mu_{12}(g_1) = u_1 + \frac{h_2}{u_1}$ for example. Then one can extend the Weyl group actions bi-rationally including generic variables $(f, g) \in \mathbb{P}^1 \times \mathbb{P}^1$. The nontrivial actions are as follows:

$$c(f) = g, \quad c(g) = f, \quad \mu_{ij}(f) = \tilde{f}, \quad \nu_{ij}(g) = \tilde{g}, \quad (42)$$

where, \tilde{f} and \tilde{g} are rational functions in (f, g) defined by

$$\begin{aligned} \frac{\tilde{f} - \mu_{ij}(f_i)}{\tilde{f} - \mu_{ij}(f_j)} &= \frac{(f - f_i)(g - g_j)}{(f - f_j)(g - g_i)}, \\ \frac{\tilde{g} - \nu_{ij}(g_i)}{\tilde{g} - \nu_{ij}(g_j)} &= \frac{(g - g_i)(f - f_j)}{(g - g_j)(f - f_i)}. \end{aligned} \quad (43)$$

Let r and T_1 be the following compositions

$$r = s_{12}\mu_{12}s_{34}\mu_{34}s_{56}\mu_{56}s_{78}\mu_{78}, \quad T_1 = crcr. \quad (44)$$

Their actions on variables (h_i, u_i, f, g) are given by

$$\begin{aligned} r(h_1) &= v h_2, & r(h_2) &= h_2, & r(u_i) &= \frac{h_2}{u_i}, & r(f) &= \bar{f}v, & r(g) &= g, \\ T_1(h_1) &= \frac{h_1}{q}v^2, & T_1(h_2) &= q h_2 v^2, & T_1(u_i) &= u_i v, & T_1(f) &= \bar{f}v, & T_1(g) &= \bar{g}v, \end{aligned} \quad (45)$$

where $v = q h_2 / h_1$. Hence, the evolution T of q -Painlevé equation (7) is the affine Weyl group translation T_1 up to re-scaling of the parameters and variables.

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